THE BUSY PERIOD ANALYSIS OF A MULTICHANNEL MARKOVIAN QUEUE STUDY AN ALTERNATIVE APPROACH

ADITYA KUMAR MISHRA

(Ph.D TMBU Bhagalpur)

aditya_12984@rediffmail.com

ABSTRACT

In this Paper, we study the busy period of an $M|M|c|\infty$ queue and obtain a closed form expression for its density function which helps in finding explicitly its moments of any desired order. Also, we discuss some special cases which can be derived easily.

Key Words:- Busy Period, Markovian queues, Mean and Variance, Laplace transform,Transient probabilities.

1.1 Introduction

Ever since Borel (1942) introduced the concept of busy period. , it has almost become an integral part of the study of any queueing situation, as it helps the planning and efficient execution of the system. It has, therefore, attracted the attention of many researchers which has resulted in some innovate ideas and techniques to derive the density function of the length of the busy period and its parameters, mostly mean and variance.

For one channel a busy period is defined to begin with the arrival of a customer to an idle channel and to end when the channel next becomes idle. In an analogous fashion, let us define a k-channel busy period for a $M|M|c|\infty$ queueing system ($0 \le k \le c$) as the time from the instant of arrival of a unit that makes *k* channels busy to the first subsequent instant when one of the *k* channels becomes empty.

The investigation of the busy period of the Markovian queues was initiated by Palm (1943, 1947) and later followed by many researchers. Bailey (1954) discussed the busy period using the generating functions. Takâcs (1962) derived the Laplace transform of the busy period and also obtained an explicit form of its expected length. Kendall (1951) obtained the length of the busy period for the general service time distribution through an interesting application of a branching process. Marlin and McGregor (1958) investigated the busy period of many server queueing systems with Poisson input and exponential service times by applying the theory of Stieljes moment problem. Neuts (1964) obtained the joint distribution of the length of busy period and the maximum number of customers present during this period by the birth and death technique based on absorbing states. The busy period for the single server queue with Poisson input ($M|M|1$ queue) is studied by Erlander (1965). Rice (1962) discussed the busy period of single server systems. Conolly (1971, 1974) studied the busy period by direct arguments in the time domain, using the Laplace transform of the generating function of the joint distribution of the length of the busy period and the number of customers served during it when one customer is present initially; obtained a formula for the Laplace transform of the joint probability of the total number of customers served during a busy period and the duration of the busy period. Bunday and El-Badri (1985) investigated the busy period for the $M|M|1$ machine interference model and derived numerical and Laplace transform solutions for the transient probabilities and duration of the operative's busy period. Generating functions for the number of repairs completed in a busy period are obtained and explicit expressions for the mean and variance of this distribution are calculated. In addition the distribution of the minimum number of machines running during a busy period is derived in explicit form. Using generating function technique and the properties of Bessel functions, the distribution of the busy period for multi-server queue is obtained by Sharaf Ali and Parthasarathy (1989). Sharma and Shobha (1986) studied the busy period of an *M*|*M*|1|*N* queue. The expression for its density function is obtained in closed form. It is shown that the density function is a mixture of exponential distributions. Also its important parameters have been worked out. Abate and Whit (1988) considered different kinds of approximations in a simple closed form and discussed their accuracy for the cumulative distribution function of the busy period for single server queue with poisson input. Le NY et. al. (1991) deal with the computation of the busy period distribution of the $M|M|1$ queue. A simple expression of this distribution without any use of transforms or Bessel functions is obtained. Recently, Sharma (1990) proposed a two dimensional state model technique to study Markovian queues and its applicability is demonstrated by means of certain examples. Shobha (1985) obtained the busy period density functions (initiated by the first customer) for the finite capacity IM queues, $M|M|1|N$ and $M|M|2|N$. The expressions obtained can be easily used to derive the various important parameters of the distribution. Sharma and Dass (1989) studied the analysis for the initial busy period generated by k servers for an $M|M|c|N$ queue with an arbitrary number of customers present initially using matrix technique. Sharma and Maheshwar (1993) considered the two dimensional state model technique developed by the first author to obtain an alternative series form expression for the busy period density function of the $M|M|1$ queue. Vatsla applied the same technique (i.e., the two dimensional state model technique) on Chapman-Kolmogorov equations of an $M|M|c|N$ queue to calculate the busy period density function by taking an arbitrary "k" number of units present in the system at time t = 0 and obtained the general form for the p^{th} order moment of the busy period distribution.

In this Paper, we study the busy period of an $M|M|c|\infty$ queue and obtain a closed form expression for its density function which helps in finding explicitly its moments of any desired order. Also, we discuss some special cases which can be derived easily.

1.2 The Model and its Analysis

Consider the usual multichannel Markovian queueing system with arrival rate λ and consisting of *c* counters with service rate at each counter being μ so that the traffic intensity is $\rho = \lambda/c\mu$. The length of ordinary busy period *T* of the system begins with the presence of *c* units which makes all the *c* servers busy and ends when one of the *c* servers become free next.

Let Q(t) be the number of units in the system at time t, and

$$q_n(t) = pr\,\{Q(t) = n\,|Q\,(0) = c\}. \qquad (1.2.1)$$

Obviously, the c - 1[th] state is an absorbing state and if b(t) and B(t) represent the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) respectively of the length of the busy period T , then

$$b(t) = \frac{d}{dt} q_{c-1}(t) \text{ and } B(t) = q_{c-1}(t) \qquad (1.2.2)$$

and the governing equations of the systems are given by

$$q'_{c-1}(\tau) = q_c(\tau) \quad (\because \text{ of the absorbing barrier})$$
$$q'_c(\tau) = -(1 + \rho)q_c(\tau) + q_{c+1}(\tau)$$
$$q'_c(\tau) = -(1 + \rho)q_n(\tau) + q_{n-1}(\tau) + q_{n+1}(\tau), n > c \qquad (1.2.3)$$

with $q'_n(\tau) = \frac{d}{d\tau} q_n(\tau)$ and $q_c(0) = \delta_{cn}$ and the time has been scaled such that $\tau = c\mu t$.

Ignoring the first equation it is assumed that the system (1.2.3) has a solution of the form

$$q_n(\tau) = e^{-(1+\rho)\tau}\rho^{n-c}\sum_{m=0}^{\infty} a(m,n)\frac{\tau^m}{m!} \qquad (1.2.4)$$

Substituting (1.2.4) in (1.2.3), we get the recurrence relations

$$a(m + 1, c) = \rho a(m, c + 1) \qquad (1.2.5)$$

$$a(m + 1, n) = a(m, n— 1) + \rho a(m, n + 1), n > c \qquad (1.2.6)$$

with $a(0, c) = 1$ & $a(0, n) = 0\ \forall\, n \neq c.$

Theorem 1.2.1 *For non-negative integers m and n, n > c*

$$a(m,n) = \begin{cases} 0 & , m < n - c \\ \left\{\left(\binom{m}{\left[\frac{m+c-n}{2}\right]}\right) - \left(\binom{m}{\left[\frac{m+c-n-1}{2}\right]}\right)\right\} & , m \geq n - c \end{cases} \qquad (1.2.7)$$

where [x] stands for the integral value of x when; it exists.

Proof : The theorem is proved by induction on in.

For m = 0 we have

$$a(0, c) = 1 \text{ and } a(0, n) = 0, \forall\, n = c + 1, c + 2, \ldots$$

For m = 1, using (1.2.7) we get

$$a(1, c) = 0$$

$$= \rho.0 = \rho.a(0, c+1)$$

Thus (1.2.5) holds for m = 0, 1.

Similarly, from (1.2.6) for m = 0, n = c. we obtain

R.H.S. = $a(0,c) + \rho a(0, c+2)$

$= 1 + \rho.0 = 1$

$= a(1, c+\ 1)$= L.H.S

Applying (1.2.7) on (1.2.6) with n = c + 2, c + 3,…, we get

R.H.S. = 0 = L.H.S.

i.e. $a(1,n) =\ 0, \forall n = c+2, c+3, ...$

Working likewise for m = 2 and n = c, we can obtain

$$a(2,c) = \left\{\binom{2}{1} - \binom{2}{0}\right\}\rho$$

$$= \ \rho.1 = \rho.a(1, c+1)$$

$$\therefore a(2,c) =\ \rho.a(1, c+1)$$

and for $n\ =\ c +\ 1$, using (1.2.6) we get

R.H.S. = $a(1,c)\ +\ \rho a(1, c\ +\ 2)$

$= 0 = a(2, c\ +\ 1)\ =\ L.H.S.$

$$\therefore a(2, c+1) =\ a(1,c) + \rho a(1, c+2)$$

For $n\ =\ c+2$, (1.2.6) gives

R.H.S. = $a(1, c+1) + \rho a(1, c+3)$

$= 1 + \rho.O = a(2, c+2)$

= L.H.S.

For $n = c+3, c+4, ...$ we get

R.H.S. = 0 = L.H.S.

which shows that (1.2.6) holds for $m = 0, 1{,}2$. Thus the theorem holds for $m = 0{,}1{,}2; n\ \geq c$and let it be

true for some general in (say).

Now if m is even, say 2*l* then

$$\left[\frac{m}{2}\right] = \left[\frac{m+1}{2}\right] = l \text{ and } \left[\frac{m+1}{2}\right] = l-1$$

also, since (1.2.7) is true for some *m*, whence we get

$$a(m+1, c) = \left\{\binom{2l}{l} - \binom{2l}{l}\right\}\rho^l = 0$$

$$a(m, c+1) = \left\{\binom{2l}{l} - \binom{2l}{l}\right\}\rho^l = 0$$

$$\therefore a(m+1, c) = 0 = \rho a(m, c+1),$$

and if m is odd say *2 l* + 1, then

$$\left[\frac{m}{2}\right] = \left[\frac{m-1}{2}\right] = l \text{ , } \left[\frac{m+1}{2}\right] = l + 1 \text{ and } \left[\frac{m-2}{2}\right] = l - 1,$$

which implies that

$$a(m+1,c) = \left\{\binom{2l+2}{l+1} - \binom{2l+2}{l}\right\}\rho^{l+1}$$

$$= \left\{\binom{2l+1}{l+1} - \binom{2l+1}{l} - \binom{2l+1}{l} - \binom{2l+1}{l+1}\right\}\rho^{l+1}$$

$$= \left\{\binom{2l+1}{l+1} - \binom{2l+1}{l-1}\right\}\rho^{l+1}$$

$$= \left\{\binom{m}{\left[\frac{m-1}{2}\right]} - \binom{m}{\left[\frac{m-2}{2}\right]}\right\}\rho.\rho^{\left[\frac{m-1}{1}\right]} \quad \left(using \binom{m}{r} = \binom{m}{m-r}\right)$$

$$= \rho a(m, c+1).$$

Also for $n > c$

$$a(m,n-1) + \rho a(m,n+1) = \left\{\binom{m}{\left[\frac{m+c-n+1}{2}\right]} - \binom{m}{\left[\frac{m+c-n}{2}\right]}\right\}\rho^{\left[\frac{m+c-n+1}{2}\right]} +$$

$$\rho\left\{\binom{m}{\left[\frac{m+c-n-1}{2}\right]} - \binom{m}{\left[\frac{m+c-n-2}{2}\right]}\right\}\rho^{\left[\frac{m+c-n-1}{2}\right]}$$

Now if $m + c - n$ is even, say 2l, then

$$\left[\frac{m+c-n}{2}\right] = \left[\frac{m+c-n+1}{2}\right] = l$$

$$\left[\frac{m+c-n+2}{2}\right] = l + 1 \text{ and } \left[\frac{m+c-n-1}{2}\right] = l - 1$$

which yields

$$a(m,n-1) + \rho a(m,n+1) = \left\{\binom{2l}{l} - \binom{2l}{l} + \binom{2l}{l-1} - \binom{2l}{l-1}\right\}\rho^{l}$$

$$= 0 = a(m+1,n),$$

and if $m + c - n$ is odd, say $2l + 1$, then

$$\left[\frac{m+c-n}{2}\right] = \left[\frac{m+c-n-1}{2}\right] = l$$

$$\left[\frac{m+c-n+1}{2}\right] = \left[\frac{m+c-n+2}{2}\right] = l + 1$$

$$\therefore a(m,n-1) + \rho a(m,n+1) = \left\{\binom{2l+1}{l+1} - \binom{2l+1}{l} + \binom{2l+1}{l} - \binom{2l+1}{l-1}\right\}\rho^{l+1}$$

$$= \left\{\binom{2l+2}{l+1} - \binom{2l+2}{l}\right\}\rho^{l+1}$$

$$= \left\{\binom{m+1}{\left[\frac{m+c-n+1}{2}\right]} - \binom{m+1}{\left[\frac{m+c-n}{2}\right]}\right\}\rho^{\left[\frac{m+c-n+1}{2}\right]}$$

$$= a\,(m+1,n)$$

which establishes the theorem.

Hence from (1.2.4), we obtain

$$q_n(t) = e^{-(\lambda+c\mu)t}\rho^{n-c}\sum_{m=0}^{\infty}\left\{\binom{m}{\left[\frac{m+c-n}{2}\right]} - \binom{m}{\left[\frac{m+c-n-1}{2}\right]}\right\}\rho^{\left[\frac{m+c-n}{2}\right]}\frac{(c\mu t)^m}{m!}\,, n \geq c$$

so that from (1.2.2), the p.d.f. of the busy period is given by

$$b(t) = c\mu q_c(t)$$

$$= c\mu e^{-(\lambda+c\mu)t}\sum_{m=0}^{\infty}\left\{\binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]}\right\}\rho^{\left[\frac{m}{2}\right]}\frac{(c\mu t)^m}{m!} \quad (1.2.8)$$

To prove that this formula is equivalent to that mentioned in Saaty (1961). Clearly, the coefficient of t^min (1.2.8) will vanish if *m* is odd and when *m* is even, say $2k$, then $b(t)$ is given by

$$b(t) = c\mu e^{-(\lambda+c\mu)t}\sum_{k=0}^{\infty}\left\{\binom{2k}{k} - \binom{2k}{k-1}\right\}\rho^k\frac{(c\mu t)^{2k}}{2k!}$$

$$= c\mu e^{-(\lambda+c\mu)t}\sum_{k=0}^{\infty}\frac{(c\lambda\mu)^k}{k!\,(k+1)}\,t^{2k}$$

which agrees with (1.93) of Saaty (1961).

1.3 Other Results

We first establish two theorems which shall later be used in deriving some important results.

Theorem 1.3.1 For non-negative integer k and real $x \neq 1$,

$$\sum_{n=0}^{\infty}\binom{2n+k}{n}\frac{x^n}{(1+x)^{2n}} = \frac{(1+x)^{k+1}}{1-x} \qquad (1.3.1)$$

Proof: We prove the result by induction on *k*. For *k* = 1, we have

$$\sum_{n=0}^{\infty}\binom{2n+k}{n}\frac{x^n}{(1+x)^{2n}}=\sum_{n=0}^{\infty}\frac{(2n+1)!}{n!\,(n+1)!}\left\{\frac{x}{(1+x)^2}\right\}^n$$

$$=\sum_{n=0}^{\infty}\frac{(2n+1).(2n-1)\cdots 7\cdot 5\cdot 3\cdot 2l\cdot(2l-2)\cdot 6\cdot 4\cdot 2}{(n+1)n\cdot 3\cdot 2\cdot 1\cdot n!}\left\{\frac{x}{(1+x)^2}\right\}^n$$

$$=\sum_{n=0}^{\infty}\frac{2^n\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\cdot\frac{2n+1}{2}\cdot 2^n\cdot 1\cdot 2\cdots\cdot n}{2\cdot 3\cdot 4\cdots(n+1)\cdot n!}\left\{\frac{x}{(1+x)^2}\right\}^n$$

$$=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\cdot(1)_n}{(2)_n\cdot n!}\left\{\frac{4x}{(1+x)^2}\right\}^n \text{ where } (a)_m=a(a+1)\ldots(a+m-1)$$

$$= {}_2F_1\left(\frac{3}{2},1,2,\frac{4x}{(1+x)^2}\right) \qquad (1.3.2)$$

where ${}_2F_1$ is the confluent hypergeometric function given by

$${}_2F_1(a,b,c,x)=\sum_{n=0}^{\infty}\frac{(a)_m\cdot(b)_m}{(c)_m\cdot m!}$$

$$=\frac{1}{B(b,c-b)}\int_0^1(1-t)^{c-b-1}\,t^{b-1}(1-xt)^{-a}dt$$

and

$$B(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Thus from (1.3.2), we obtain

$$\sum_{n=0}^{\infty}\binom{2n+1}{n}\frac{x^n}{(1+x)^{2n}}=\frac{1}{B(1,1)}\int_0^1\left(1-\frac{4xT}{(1+x)^2}\right)^{-\frac{3}{2}}dt$$

$$=\frac{(1+x)^2}{2x}\left[\frac{1+x}{1-x}-1\right]=\frac{(1+x)^2}{1-x}$$

which establishes the results for k = 1. Now let the theorem be true for some k = m (say), i.e.

$$\sum_{n=0}^{\infty}\binom{2n+m}{n}\frac{x^n}{(1+x)^{2n}}=\frac{(1+x)^{m+1}}{1-x}$$

then for k = m + 1, we have

$$\sum_{n=0}^{\infty}\binom{2n+m+1}{n}\frac{x^n}{(1+x)^{2n}}$$

$$=\sum_{n=0}^{\infty}\left\{\binom{2n+m+2}{n+1}-\binom{2n+m+1}{n+1}\right\}\frac{x^n}{(1+x)^{2n}}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{(2n+m+2)!}{(n+1)!\,(n+m-1)!} - \frac{(2n+m+1)!}{(n+1)!\,(n+m)!} \right\} \frac{x^n}{(1+x)^{2n}}$$

$$= \frac{(1+x)^2}{x} \sum_{n=1}^{\infty} \left\{ \binom{2n+m}{n} - \binom{2n+m+1}{n} \right\} \frac{x^n}{(1+x)^{2n}}$$

$$= \frac{(1+x)^2}{x^n} \left\{ \sum_{n=0}^{\infty} \binom{2n+m}{n} \frac{x^n}{(1+x)^{2n}} - \sum_{n=0}^{\infty} \binom{2n+m-1}{n} \frac{x^n}{(1+x)^{2n}} \right\}$$

$$= \frac{(1+x)^2}{x^n} \left\{ \frac{(1+x)^{m+1}}{1-x} - \frac{(1+x)^m}{1-x} \right\} = \frac{(1+x)^m + 2}{1-x}$$

Hence the result is true for all k

Theorem 1.3.2 For a non-negative integer k and real $x \neq 1$,

$$\sum_{n=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \binom{m+k}{m} \left(\frac{x}{(1+x)^2} \right)^{\left[\frac{m}{2}\right]} =$$

$$\begin{cases} 1+x & ,k=0 \\ \dfrac{(1+x)^2}{1-x} & ,k=1 \\ \dfrac{(1+x)^{k+1}}{(1-x)^{2k-1}} \sum_{j=0}^{k-2} \binom{k-1}{j} \binom{k-2}{j} \dfrac{x^j}{j+1} & ,k \geq 2 \end{cases} \qquad (1.3.3)$$

Proof: The theorem is proved by induction on *k* for *k* = 0,

L.H.S.

$$\sum_{m=0}^{k-2} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \left\{ \frac{x}{(1+x)^2} \right\}^{\left[\frac{m}{2}\right]}$$

$$= \sum_{m=0}^{\infty} \frac{2m!}{m!\,(m+1)!} \left(\frac{x}{(1+x)^2} \right)^m$$

$$= \sum_{m=0}^{\infty} \frac{2m!}{m!\,(m+1)!} \left(\frac{x}{(1+x)^2} \right)^m - 2 \sum_{m=1}^{\infty} \frac{2m!}{(m-1)!\,(m+1)!} \left(\frac{x}{(1+x)^2} \right)^m \quad (4.3.4)$$

Applying Theorem 1.3.1 on (1.3.4) by taking k = 1 and 2, one finds

$$L.H.S. = \frac{(1+x)^2}{1-x} - \frac{2x(1+x)}{1-x} = 1+x \qquad (1.3.5)$$

Again for k = 1, L.H.S. of (1.3.3) easily works out to

$$L.H.S. = \sum_{m=0}^{\infty} \frac{(2m+1)!}{m!\,(m+1)!} \left\{ \frac{x}{(1+x)^2} \right\}^2 = \frac{(1+x)^2}{1-x}, (\because of\ (3.3.1)) \quad (1.3.6)$$

And for k = 2, L.H.S. of (1.3.3) gives -

$$L.H.S. = \sum_{n=0}^{\infty}\left\{\binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]}\right\}\left\{\frac{x}{(1+x)^2}\right\}^{\left[\frac{m}{2}\right]}\frac{(m+2)(m+1)}{2\times 1}$$

$$= \sum_{m=0}^{\infty}\frac{(2m+2)!}{m!\,(m+1)!}\left\{\frac{x}{(1+x)^2}\right\}^m$$

$$= \sum_{m=0}^{\infty}\frac{(2m+1)!}{m!\,(m+1)!}\left(\frac{x}{(1+x)^2}\right)^m + \sum_{m=1}^{\infty}\frac{(2m+1)!}{(m-1)!\,(m+1)!}\left\{\frac{x}{(1+x)^2}\right\}^m \quad (1.3.7)$$

Now taking k = 1 in (1.3.1) and differentiating both sides w.r.t. x and simplifying, we get

$$\sum_{n=0}^{\infty}\binom{2n+1}{n} - \left\{\frac{x}{(1+x)^2}\right\}^n = \frac{x(1+x)^2(3-x)}{(1-x)^3} \quad (1.3.8)$$

and using (1.3.8) in (1.3.7), we finally get

$$L.H.S. = \frac{(1+x)^2}{1-x} + \frac{x(1+x)^2(3-x)}{(1-x)^3}$$

$$= \frac{(1+x)^3}{(1-x)^3}$$

which establishes (1.3.3) for k = 0,1 and 2. Thus for some k, k = n, we have

$$\sum_{n=0}^{\infty}\left\{\binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]}\right\}\left\{\frac{x}{(1+x)^2}\right\}^{\left[\frac{m}{2}\right]}$$

$$= \frac{(1+x)^{n+1}}{(1-x)^{2n-1}}\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\frac{x^j}{j+1}, n \geq 2 \quad (1.3.9)$$

and, therefore, for k = n + 1, we have

$$L.H.S. = \sum_{m=0}^{\infty}\frac{(2m+n+1)!}{m!\,(m+1)!\,(n+1)!}\left\{\frac{x}{(1+x)^2}\right\}^m$$

$$= 2\sum_{m=0}^{\infty}\frac{m!\,(2m+n)!}{m!\,(m+1)!\,(n+1)!}\left\{\frac{x}{(1+x)^2}\right\}^m + \sum_{m=0}^{\infty}\frac{(2m+n)!}{m!\,(m+1)!\,n!}\left\{\frac{x}{(1+x)^2}\right\}^m \quad (1.3.10)$$

Now, differentiating (1.3.9) w.r.t. *x* on both sides and rearranging, we get

$$\sum_{m=0}^{\infty}\frac{m(2m+n)!}{m!\,(m+1)!\,n!}\left\{\frac{x}{(1+x)^2}\right\}^m = \frac{x(1+x)^{n+1}}{(1-x)^{2n-1}}\{3n + x(n-2)\}\times$$

$$\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\frac{x^j}{j+1} + \frac{(1+x)^{n+2}}{(1-x)^{2n}}\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\frac{jx^j}{j+1}$$

$$= \frac{(1+x)^{n+1}}{(1-x)^{2n+1}}\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\{(n-j-2)x^2 + 3nx + j\}\frac{x^j}{j+1} \quad (1.3.11)$$

and making use of (1.3.11) in (1.3.10) and after some algebra we easily obtain

$$L.H.S.=\frac{2}{n+1}\cdot\frac{(1+x)^{n+1}}{(1-x)^{2n+1}}\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\{(n-j-2)x^2+3nx+j\}\frac{x^j}{j+1}$$

$$+\frac{(1+x)^{n+1}}{(1-x)^{2n-1}}\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\frac{x^j}{j+1}$$

$$=\frac{(1+x)^{n+2}}{(1-x)^{2n+1}}\left\{\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\frac{2(n-j-2)x^2+6nx+2j}{(n+1)(1+x)}\cdot\frac{x^j}{j+1}\right.$$

$$\left.+\frac{(1-x)^{n+1}}{1+x}\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\frac{x^j}{j+1}\right\}$$

$$=\frac{(1+x)^{n+2}}{(1-x)^{2n+1}}\left\{\sum_{j=0}^{n-2}\binom{n-1}{j}\times\binom{n-2}{j}\cdot\frac{x^2(3n-2j-3)+2x(2n-1)+n+2j+1}{(1+x)(n+1)}\cdot\frac{x^j}{j+1}\right\}$$

$$=\frac{(1+x)^{n+2}}{(1-x)^{2n+1}}\left\{\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}\cdot\frac{(3n-2j-3)+x+n+2j+1}{(n+1)}\cdot\frac{x^j}{j+1}\right\}$$

$$=\frac{(1+x)^{n+2}}{(n+1)(1-x)^{2n+1}}\left\{\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}(3n-2j-3)\frac{x^{j+1}}{j+1}+\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}(n+2j\right.$$

$$\left.+1)\frac{x^j}{j+1}\right\}$$

Now, replacing j by $j-1$ in the first sum, we can easily get

$$L.H.S=\frac{(1+x)^{n+2}}{(n+1)(1-x)^{2n+1}}\left\{\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}(3n-2j-3)\frac{x^j}{j}+\sum_{j=0}^{n-2}\binom{n-1}{j}\binom{n-2}{j}(n+2j\right.$$

$$\left.+1)\frac{x^j}{j+1}\right\}$$

$$=\frac{(1+x)^{n+2}}{(n+1)(1-x)^{2n+1}}\sum_{j=0}^{n-1}\left\{\frac{j(3n-2j-1)}{n(n-1)(n+1)}\binom{n}{j}\binom{n-1}{j}+\frac{(n+2j+1)\,(n-j)(n-j-1)}{n(n-1)(n+1)(j+1)}\binom{n}{j}\binom{n-1}{j}\right\}x^j$$

$$=\frac{(1+x)^{n+2}}{(n+1)(1-x)^{2n+1}}\sum_{j=0}^{n-1}\frac{1}{n(n-1)(n+1)}\binom{n}{j}\binom{n-1}{j}\frac{x^j}{j+1}$$

$$\times\{(3n-2j-1)(j+1)j+(n+2j+1)(n-j)(n-j-1)\}$$

$$= \frac{(1+x)^{n+2}}{(1-x)^{2n+1}} \sum_{j=0}^{n-1} \binom{n}{j} \binom{n-1}{j} \frac{x^j}{j+1}$$

which establishes the result for all k > 2.

We now show that b(t) is. a va1iddensity function. We have

$$\int_0^\infty b(t)dt = c\mu \sum_{m=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \rho^{\left[\frac{m}{2}\right]} \frac{c\mu^m}{m!} \times \int_0^\infty e^{-(\lambda+c\mu)t} t^m dt$$

$$= \sum_{m=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \frac{m!\, c\mu^m \rho^{\left[\frac{m}{2}\right]}}{(\lambda+c\mu)^{m+1} m!}$$

$$= \frac{1}{(1+\rho)} \sum_{m=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \left(\frac{\rho}{1+\rho}\right)^{\left[\frac{m}{2}\right]}$$

Using Theorem 1.3.2 with k = 0, x = ρ, we get

$$\int_0^\infty b(t)dt = \frac{1+\rho}{1+\rho}$$

$$= 1$$

1.4 Moments

The k^{th} order moment of the length of the c-channel busy period T_c is given by

$$E(T_c^k) = \int_0^\infty t^x b(t)dt \qquad (1.4.1)$$

$$= \sum_{m=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \rho^{\left[\frac{m}{2}\right]} \frac{c\mu^{m+1}}{m!} \times \int_0^\infty e^{-(\lambda+c\mu)t} t^{m+k} dt$$

$$= \sum_{m=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \frac{\rho^{\left[\frac{m}{2}\right]} c\mu^{m+1}}{(\lambda+c\mu)^{m+k+1}} \frac{(m+k)!}{m!}$$

$$= \frac{1}{(c\mu)^k (1+\rho)^{k+1}} \sum_{m=0}^{\infty} \left\{ \binom{m}{\left[\frac{m}{2}\right]} - \binom{m}{\left[\frac{m-1}{2}\right]} \right\} \left(\frac{\rho}{1+\rho}\right)^{\left[\frac{m}{2}\right]} \frac{(m+k)!}{m!}$$

Using Theorem 1.3.2, we easily obtain

$$E(T_c^k) = \begin{cases} \dfrac{1}{(c\mu)^k (1-\rho)} & , k = 1 \\ \dfrac{k!}{(c\mu)^k (1-\rho)^{2k-1}} \sum_{j=0}^{k-2} \binom{k-1}{j} \binom{k-2}{j} \dfrac{\rho^j}{j+1}, & k \geq 2 \end{cases} \qquad (1.4.2)$$

__**Particular cases :**__

Case 1 : $c = k = 1$, $\rho = \frac{\lambda}{\mu}$

In this case we have M|M|1|∞ queue and we get

$$b(t) = \mu e^{-(\lambda+\mu)t} \sum_{m=0}^{\infty} \left\{ \left(\binom{m}{\left[\frac{m}{2}\right]} \right) - \left(\binom{m}{\left[\frac{m-1}{2}\right]} \right) \right\} \rho^{\left[\frac{m}{2}\right]} \frac{(\mu t)^m}{m!}$$

$E(T) = \frac{1}{\mu-\lambda}$,

$E(T) = \frac{2}{\mu^2(1-\rho)^3}$ and $V(T) = \frac{1+\rho}{\mu^2(1-\rho)^3}$

Case 2: $c = k = 2, \rho = \frac{\lambda}{(2\mu)}$.

This represents the M|M|2|∞ queue and we obtain

$$b(t) = 2\mu e^{-(\lambda+\mu)t} \sum_{m=0}^{\infty} \left\{ \left(\binom{m}{\left[\frac{m}{2}\right]} \right) - \left(\binom{m}{\left[\frac{m-1}{2}\right]} \right) \right\} \rho^{\left[\frac{m}{2}\right]} \frac{(2\mu t)^m}{m!}$$

$E(T) = \frac{1}{2\mu-\lambda}$,

$E(T) = \frac{1}{2\mu^2(1-\rho)^3}$ and $V(T) = \frac{1+\rho}{4\mu^2(1-\rho)^3}$ (1.4.3)

which all agree with the results in the literature.

1.5 Numerical Illustrations

In order to check the efficiency of the results obtained, numerical computations were made on the system ICL3980 to calculate the distribution function B(t) for different values of t; the values of B(t) are shown in Table 1.1 for different values of the number of servers c. The values of the k^{th} order moment of the length of the c-channel busy period are given in Table 1.2 for k = 1, 2, 3,4,5,6.

Table 1.1

The values of busy period distribution for different values of c in $M|M|c|\infty$ queue

λ = 0.4, μ = 0.4

Time	**Values of c and p**				
	c =1 p=l	**c = 2 p = 0.5**	**c = 3 p = 113**	**c = 4 p = 0.25**	**c = 5 p = 0.25**
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
1.00	0.2813524	0.4842436	0.6303253	0.7353499	09451544
2.00	0.4276655	0.6766916	0.8193795	0.9000537	0.9806491
3.00	0.5148966	0.7740431	0.8980054	0.9551079	0.9924337
4.00	0.5727525	0.8317935	0.9375856	0.9778702	0.9968513
5.00	0.6142472	0.8696866	0.9598507	0.9884506	0.9986336
6.00	0.6457381	0.8962475	0.9732300	0.9937352	0.9993888
7.00	0.6706467	0.9157351	0.9817723	0.9965054	0.9997203
8.00	0.6909707	0.9305166	0.9873291	0.9980089	0.999697
9.00	0.7079568	0.9420150	0.9910600	0.9988464	0.9999384
10.00	0.7224257	0.9511381	0.9936158	0.9993225	0.9999705
11.00	0.7349418	0.9584929	0.9953952	0.9995977	0.9999858
12.00	0.7459073	0.9645001	0.9966506	0.9997588	0.9999931

13.00	0.7556175	0.9694608	0.9975461	0.9998543	0.9999966
14.00	0.7642946	0.9735954	0.9981909	0.9999113	0.9999983
15.00	0.7721095	0.9770691	0.9986590	0.9999457	0.9999992
16.00	0.7791960	0.9800079	0.9990012	0.9999666	0.9999996
17.00	0.7856605	0.9825094	0.9992529	0.9999793	0.9999998
18.00	0.7915890	0.9846500	0.9999872	0.9999872	0.9999999
19.00	0.7970516	0.9864906	0.9995773	0.9999920	0.9999999
20.00	0.8021063	0.9880801	0.9996805	0.9999950	1.0000000
21.00	0.8068013	0.9894581	0.9997578	0.9999969	1.0000000
22.00	0.8111774	0.9906568	0.9998159	0.9999980	1.0000000
23.00	0.8152692	0.9917029	0.9998598	0.9999988	1.0000000
24.00	0.8191061	0.9926185	09998929	0.9999992	1.0000000
25.00	0.8227135	0.9934220	0.9999181	0.9999995	1.0000000

Table 1.2

The values of k^{th} order moment of the length of-the busy period T

for different values of k and c

λ = 0.4, μ = 0.4

Time	**Values of c and p**				
	c =1 **p=l**	**c = 2** **p = 0.5**	**c = 3** **p = 113**	**c = 4** **p = 0.25**	**c = 5** **p = 0.2**
1	1.66666664	1.66666664	1.11111106	0.83333332	0.66666666
2	7.40740688	7.40740688	3.29218093	1.85185172	1.18518518
3	12.34567842	12.34567842	3.65797861	1.54320980	0.79012345
4	27.43484094	27.43484094	5.41922733	1.71467755	0.70233196
5	76.20789013	76.20789013	10.03560521	2.38149656	0.78036884
6	254.02628888	254.02628888	22.30134420	3.96916076	1.04049179

References

1. Palm, C. (1943) - Intensity fluctuation in telephonic traffic, *Ericsson Tech*., 1, 85.

2. Bailey' N.T.J. (1954) - A continuous time treatment of a simple queue using generating functions, *J. Roy. Stat. Soc*., B16, 288 - 291.

3. Kendal P.C. (1951) - Some problems in the theory of queues, *J. Roy. Slat. Soc.,* B13, 151 - 185.

4 Neuts, M F (1964) The distribution of the maximum length of a Poisson queue during a busy period, *Oper. Res*., 12, 281 - 285.

5.. Marlin, S. and Mc Gregor, I. (1958) - Many server queueing processes with Poisson input and exponential service times, *Pacific J. Math*., 8, 87 - 118.

6. Erlander, S. (1965) - The remaining busy period for a single server queue with Poisson input, *Oper. Res*., 14, 444 -459.

7 .Rice, S.O. (1962) - Single server systems - 11. Busy periods, *The Bell System Tech, J*., 41, 279 - 310.

8. Sharma, O.P. and Maheshwar, M.V.R. (1993) - A note on the alternative form of the busy period density of an $M|M|1$ queue, *J. Stoch. Anal. Appl.,* 11(2), 231 - 234.

9. Sharaf Ali, M. and Parthasarathy, P.R. (1989) - On the distribution of a busy period for the many server Poisson queue, *Opsearch*, 26,125 - 132.

10. Le NY, L.M., Rubino, G. and Sericala, B. (1991) - Calculating the busy period distribution of the $M|M|1$ queue, *Rapporis de Recherche' No. 1501* Instial National de Recherche'en Informatique et en Automatique, France.