On Coefficient Estimates with respect to Negative Coefficients on Uniformly Starlike and Uniformly Convex Functions

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Abstract

In this paper, the authors obtain a new subclass of strongly starlike and convex functions defined by Komatu integral transforms and discussed the inclusion properties of these classes. Further, we introduce a new subclass of uniformly starlike functions and uniformly convex functions with negative coefficients defined by the Komatu integral transforms. Few properties of these classes are derived in the form of coefficient estimates, distortion and covering theorems, starlikeness and convexity.

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1 Introduction

Let $\mathcal{A}$ denotes the class of functions of the form :

$$\varphi(z) = z + \sum_{r=2}^{\infty} a_r z^r \tag{1}$$

which are *regular* in the unit disk $\Theta = \{z : |z| < 1\}$.

Let $\mathcal{S}$ be the class of functions $\varphi \in \mathcal{A}$ *regular* and *univalent* in Θ, normalized by the suitable $\varphi(0) = 0$ and $\varphi'(0) = 1$.

Let $\mathcal{T}$ denotes the class of functions of the form :

$$\varphi(z) = z - \sum_{r=2}^{\infty} a_r z^r, \quad a_r \geq 0, \quad \forall \quad r \geq 2, \tag{2}$$

which are *regular* and *univalent* in the unit disk $\Theta = \{z : |z| < 1\}$, introduced and studied by Silverman[16].

Definition 1.1. *A function $\varphi \in \mathcal{S}$ is said to be starlike, if it is univalent and $\varphi(\Theta)$is a starlike domain. Equivalently, the function $\varphi \in \mathcal{S}$ is starlike, necessary and sufficient condition for*

$$\Re\left\{\frac{z\varphi'(z)}{\varphi(z)}\right\} > 0,$$

for $z \in \Theta$ and the class of all starlike functions are denoted by $\mathcal{S}^*$.

Definition 1.2. *A function $\varphi \in \mathcal{S}$ is said to be convex, if it is univalent and $\varphi(\Theta)$ is a convex domain. Equivalently, the function $\varphi \in \mathcal{S}$ is convex, necessary and sufficient condition for*

$$\Re\left\{1 + \frac{z\varphi''(z)}{\varphi'(z)}\right\} > 0,$$

for $z \in \Theta$ and the class of all convex functions are denoted by $\mathcal{C}$.

A function $\varphi \in \mathcal{S}$ is said to be starlike(convex), if φ maps Θ conformably onto the starlike(convex) domain with respect to the origin.

Definition 1.3. *If $\varphi \in \mathcal{A}$ satisfies*

$$\left|arg\left(\frac{z\varphi'(z)}{\varphi(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta, \tag{3}$$

for some $\gamma(0 \leq \gamma < 1)$ and $(0 < \beta \leq 1)$, then $\varphi(z)$ is said to be strongly starlike of order β and type γ in Θ and is denoted by $\mathcal{S}^*(\beta, \gamma)$.

Definition 1.4. *If $\varphi \in \mathcal{A}$ satisfies*

$$\left|arg\left(1 + \frac{z\varphi''(z)}{\varphi'(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta, \tag{4}$$

for some $\gamma(0 \leq \gamma < 1)$ and $(0 < \beta \leq 1)$, then $\varphi(z)$ is said to be strongly convex of order β and type γ in Θ and is denoted by $\mathcal{C}(\beta, \gamma)$.

It is obvious that if $\varphi \in \mathcal{C}(\beta, \gamma)$ necessary and sufficient condition $z\varphi' \in \mathcal{S}^*(\beta, \gamma)$.

Definition 1.5. *[5] A function $\varphi \in \mathcal{S}$ is said to be uniformly convex in Θ necessary and sufficient condition*

$$\Re\left\{1+\frac{z\varphi''(z)}{\varphi'(z)}\right\} \geq \left|\frac{z\varphi''(z)}{\varphi'(z)}\right|, \quad (z \in \Theta)$$

and this class is denoted by $\mathcal{UCV}$.

Rønning[14] and Ramachandran et al.[11, 12] were introduced the following class of starlike functions related to $\mathcal{UCV}$ as follows:

Definition 1.6. *A function $\varphi \in \mathcal{S}$ is said to be uniformly starlike in Θ necessary and sufficient condition*

$$\Re\left\{\frac{z\varphi'(z)}{\varphi(z)}\right\} \geq \left|\frac{z\varphi''(z)}{\varphi'(z)}-1\right|, \quad (z \in \Theta)$$

and this class is denoted by $\mathcal{S}_p$.

In [13], Rønning also generalised the class $\mathcal{S}_p$ and is given below:

Definition 1.7. *A function $\varphi \in \mathcal{S}_p(\delta)$, $0 \leq \delta \leq 1$, if φ satisfies the regular characterization*

$$\Re\left\{\frac{z\varphi'(z)}{\varphi(z)}\right\} - \delta \geq \left|\frac{z\varphi''(z)}{\varphi'(z)}-1\right|, \quad (z \in \Theta)$$

and $\varphi \in \mathcal{UCV}(\delta)$ necessary and sufficient condition $z\varphi' \in \mathcal{S}_p(\delta)$.

The class $\mathcal{UCV}(\delta)$ is defined in a similar manner(see[3]).

Definition 1.8. *[7] The integral transform of $\varphi \in \mathcal{T}$ for $\alpha > -1, \sigma > 0$ is denoted by $\mathcal{I}_\alpha^\sigma \varphi(z)$ and is defined as*

$$\begin{aligned} \mathcal{I}_\alpha^\sigma \varphi(z) &= \frac{(1+\alpha)^\sigma}{z^\alpha \Gamma(\sigma)} \int_0^z (log\frac{z}{t})^{\sigma-1} t^{\alpha-1}\varphi(t)dt \\ &= z - \sum_{r=2}^{\infty} \left(\frac{1+\alpha}{r+\alpha}\right)^\sigma a_r z^r, \quad (\alpha > -1, \sigma > 0). \end{aligned} \tag{5}$$

It was introduced and studied by [7] and extended by [15].

For $\sigma = 1$, the generalized Bernardi-Libera-Livingston integral operator $\mathcal{I}_\alpha^1\varphi(z) = \mathcal{L}_\alpha\varphi(z)$ is given by

$$\begin{aligned} \mathcal{L}_\alpha\varphi(z) &= \frac{(1+\alpha)}{z^\alpha}\int_0^z t^{\alpha-1}\varphi(t)dt \\ &= z - \sum_{r=2}^{\infty}\left(\frac{1+\alpha}{r+\alpha}\right)a_r z^r, \quad (\alpha > -1), \end{aligned} \tag{6}$$

is studied by [2] and the operator $\mathcal{L}_\alpha\varphi(z)$, for $\alpha = 1, \mathcal{L}_1\varphi(z)$ was investigated by [8].

For $\alpha = 1$, [6] introduced the one parameter family of integral operator $\mathcal{I}_1^\sigma\varphi(z)$ and also see [10]

$$\begin{aligned} \mathcal{I}^\sigma\varphi(z) &= \frac{2^\sigma}{z\Gamma(\sigma)}\int_0^z (log\frac{z}{t})^{\sigma-1}\varphi(t)dt \\ &= z - \sum_{r=2}^{\infty}\left(\frac{2}{r+1}\right)^\sigma a_r z^r, \quad (\sigma > 0, \varphi \in \mathcal{A}). \end{aligned}$$

The operator $\mathcal{I}^\sigma\varphi(z)$ is deep related to the multiplier transformations studied earlier by[4].

Using the belief of Bharathi et al.[3] and Jung et al.[6], consider the class $\mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha^\sigma)$ and $\mathcal{UCVT}(\delta, \mathcal{I}_\alpha^\sigma)$.

Definition 1.9. *Let* $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha^\sigma)$, $0 \leq \delta < 1$ *be the class of functions* $\varphi \in \mathcal{T}$ *satisfying the following*

$$\left|\frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))'}{\mathcal{I}_\alpha^\sigma\varphi(z)} - 1\right| \leq \Re\left\{\frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))'}{\mathcal{I}_\alpha^\sigma\varphi(z)} - 1\right\} - \delta, \quad z \in \Theta. \tag{7}$$

Definition 1.10. *Let* $\varphi \in \mathcal{UCVT}(\delta, \mathcal{I}_\alpha^\sigma)$, $0 \leq \delta < 1$ *be the class of functions* $\varphi \in \mathcal{T}$ *satisfying the following*

$$\left|\frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))''}{(\mathcal{I}_\alpha^\sigma\varphi(z))'}\right| \leq \Re\left\{1 + \frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))''}{(\mathcal{I}_\alpha^\sigma\varphi(z))'} - \delta\right\}, \quad z \in \Theta. \tag{8}$$

Lemma 1.11. *If* $\mathcal{I}_\alpha^\sigma\varphi(z) \in \mathcal{T}$, *then*

$$\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^\sigma r a_r \leq 1.$$

Proof. Suppose

$$\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} ra_r > 1,$$

there exits an integer N such that

$$\sum_{r=2}^{N}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} ra_r > 1+\varepsilon/2$$

$\left(\frac{1}{1+\varepsilon/2}\right)^{\frac{1}{N-1}} < z < 1$, we have

$$\begin{aligned}(\mathcal{I}_{\alpha}^{\sigma}\varphi(z))' &= 1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} ra_r z^{r-1}\\ &\leq 1-\sum_{r=2}^{N}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} ra_r z^{r-1}\\ &\leq 1-z^{N-1}\sum_{r=2}^{N}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} ra_r\\ &\leq 1-z^{N-1}(1+\varepsilon/2)\\ &< 0.\end{aligned}$$

But $(\mathcal{I}_{\alpha}^{\sigma}\varphi(0))' = 1 > 0$.
There exits a real number $z_0, 0 < z_0 < 1$, such that

$$(\mathcal{I}_{\alpha}^{\sigma}\varphi(z_0)) = 0.$$

Hence $\mathcal{I}_{\alpha}^{\sigma}\varphi(z)$ is not one to one. □

Remark 1.12. *For $\alpha = 1$, Lemma 1.11 were discussed in [9].*

By taking $\sigma = 1$ Lemma 1.11, we can deduce the following:

Corollary 1.13. *Let $\varphi \in \mathcal{T}$ be given by (2). If $\mathcal{I}_{\alpha} \in \mathcal{T}$, then*

$$\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right) ra_r \leq 1.$$

The main objective of this paper is to investigate some coefficient estimates, distortion bounds for the function classes $\mathcal{S}_p\mathcal{T}(\delta,\mathcal{I}_{\alpha}^{\sigma})$ and $\mathcal{UCVT}(\delta,\mathcal{I}_{\alpha}^{\sigma})$.

2 Characterization Theorem

Using the tool adopted by [1] to analysis the coefficient estimates for the function classes $\mathcal{S}_p\mathcal{T}(\delta,\mathcal{I}_\alpha^\sigma)$ and $\mathcal{UCVT}(\delta,\mathcal{I}_\alpha^\sigma)$. The main characterization theorem for this function classes are as follows:

Theorem 2.1. *A function $\varphi \in \mathcal{S}_p\mathcal{T}(\delta,\mathcal{I}_\alpha^\sigma)$ necessary and sufficient condition*

$$\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(2r-1-\delta)a_r \leq 1-\delta, \tag{9}$$

for some $0 \leq \delta < 1$ and $\sigma > 0$. The result is sharp

$$\mathcal{I}_\alpha^\sigma\varphi(z) = z - \frac{1-\delta}{\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(2r-1-\delta)}z^r, \quad r \geq 2. \tag{10}$$

Proof.

$$\begin{aligned}\left|\frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))'}{\mathcal{I}_\alpha^\sigma\varphi(z)}-1\right| - \Re\left\{\frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))'}{\mathcal{I}_\alpha^\sigma\varphi(z)}-1\right\} &\leq 2\left|\frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))'}{\mathcal{I}_\alpha^\sigma\varphi(z)}-1\right| \\ &\leq \left|\frac{z\left(1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} r a_r z^{r-1}\right)}{z-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r z^n}-1\right| \\ &\leq \frac{\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(r-1)a_r}{1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r}.\end{aligned}$$

if (7) holds, then we have,

$$\left|\frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))'}{\mathcal{I}_\alpha^\sigma\varphi(z)}-1\right| - \Re\left\{\frac{z(\mathcal{I}_\alpha^\sigma\varphi(z))'}{\mathcal{I}_\alpha^\sigma\varphi(z)}-1\right\} \leq 1-\alpha$$

which is equivalent to (7).
Conversely, if $\varphi \in \mathcal{S}_p\mathcal{T}(\delta,\mathcal{I}_\alpha^\sigma)$,

$$\frac{1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} r a_r z^{r-1}}{1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r z^{r}} - \delta \geq \frac{\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(r-1))a_r z^{r-1}}{1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r z^{r-1}}.$$

Let $z \to 1^-$ along the real line then we have,

$$\frac{1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} r a_r}{1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r} - \frac{\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(r-1)a_r}{1-\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r} \geq \delta$$

or

$$\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r\,(2r-1-\delta)\leq \delta-1$$

which is the required result. □

Since the proof of the Characterization theorem for the class $\mathcal{UCVT}(\delta,\mathcal{I}_{\alpha}^{\sigma})$. is same the proof of Theorem 2.1, it will be deleted.

Theorem 2.2. *Let the function* $\varphi\in\mathcal{T}$. *A function* $f\in\mathcal{UCVT}(\delta,\mathcal{I}_{\alpha}^{\sigma})$ *necessary and sufficient condition*

$$\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(2r-1-\delta)ra_r\leq 1-\delta,$$

for some $0\leq\delta<1$ *and* $\sigma>0$. *The result is sharp*

$$\mathcal{I}_{\alpha}^{\sigma}\varphi(z)=z-\frac{1-\delta}{\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} r\,(2r-1-\delta)}z^r,\quad r\geq 2. \tag{11}$$

Coefficient estimates for the class $\mathcal{S}_p\mathcal{T}(\delta,\mathcal{I}_{\alpha}^{\sigma})$ and $\mathcal{UCVT}(\delta,\mathcal{I}_{\alpha}^{\sigma})$ is an immediate consequence of Theorem 2.1 and Theorem 2.2 respectively which is given below:

Theorem 2.3. *If* $\varphi\in\mathcal{S}_p\mathcal{T}(\delta,\mathcal{I}_{\alpha}^{\sigma})$, *then*

$$a_r\leq\frac{1-\delta}{\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(2r-1-\delta)},\quad r\geq 2.$$

Proof. Since $\varphi\in\mathcal{S}_p\mathcal{T}(\delta,\mathcal{I}_{\alpha}^{\sigma})$ we have,

$$\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(2r-1-\delta)\,a_r\leq\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(2r-1-\delta)\,a_r\leq 1-\delta$$

$$a_r\leq\frac{1-\delta}{\sum\limits_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(2r-1-\delta)}$$

$$a_r\leq\frac{1-\delta}{\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma}(2r-1-\delta)},\quad r\geq 2.$$

□

Theorem 2.4. *If* $\varphi\in\mathcal{UCVT}(\delta,\mathcal{I}_{\alpha}^{\sigma})$, *then*

$$a_r\leq\frac{1-\delta}{\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} r\,(2r-1-\delta)},\quad r\geq 2.$$

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Remark 2.5. *For $\alpha = 1$, Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4 were discussed in [9].*

By taking $\sigma = 1$ in Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4, we can deduce the following:

Corollary 2.6. *Let the function $\varphi \in \mathcal{T}$. A function $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha)$ necessary and sufficient condition for*

$$\sum_{r=2}^{\infty} \left(\frac{\alpha+1}{\alpha+r}\right)(2r-1-\delta)a_r \leq 1-\delta, \tag{12}$$

for some $0 \leq \delta < 1$. The result is sharp

$$\mathcal{I}_\alpha\varphi(z) = z - \frac{1-\delta}{\left(\frac{\alpha+1}{\alpha+r}\right)(2r-1-\delta)} z^r, \quad r \geq 2. \tag{13}$$

Corollary 2.7. *Let the function $\varphi \in \mathcal{T}$. A function $\varphi \in \mathcal{UCVT}(\delta, \mathcal{I}_\alpha)$ necessary and sufficient condition for*

$$\sum_{r=2}^{\infty} \left(\frac{\alpha+1}{\alpha+r}\right)(2r-1-\delta)ra_r \leq 1-\delta, \tag{14}$$

for some $0 \leq \delta < 1$. The result is sharp

$$\mathcal{I}_\alpha\varphi(z) = z - \frac{1-\delta}{\left(\frac{\alpha+1}{\alpha+r}\right) r\,(2r-1-\delta)} z^r, \quad r \geq 2. \tag{15}$$

Corollary 2.8. *If $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha)$, then*

$$a_r \leq \frac{1-\delta}{\left(\frac{\alpha+1}{\alpha+r}\right)(2r-1-\delta)}, \quad r \geq 2.$$

3 Distortion and Covering Theorems

Theorem 3.1. *If $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha^\sigma)$ and $\mid z \mid = l < 1$, then*

$$l - \frac{1-\delta}{3-\delta} l^2 \leq \mid \mathcal{I}_\alpha^\sigma \varphi(z) \mid \leq l + \frac{1-\delta}{3-\delta} l^2.$$

Equality holds true for the functions of the form:

$$\mathcal{I}_\alpha^\sigma \varphi(z) = z - \frac{1-\delta}{3-\delta} z^2. \tag{16}$$

Proof. First, it is evident that

$$(3-\delta)\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r \leq \sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+n}\right)^{\sigma}(2r-1-\delta)\, a_r.$$

If $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha^\sigma)$, using the inequality in Theorem 3.1 we get,

$$\sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r \leq \frac{1-\delta}{3-\delta}. \tag{17}$$

From (2) with $\mid z \mid= l < 1$, we get

$$\begin{aligned}\mid \mathcal{I}_\alpha^\sigma \varphi(z) \mid &\leq l + \sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r l^r \\ &\leq l + \leq \frac{1-\delta}{3-\delta} l^2,\end{aligned}$$

and

$$\begin{aligned}\mid \mathcal{I}_\alpha^\sigma \varphi(z) \mid &\geq l - \sum_{r=2}^{\infty}\left(\frac{\alpha+1}{\alpha+r}\right)^{\sigma} a_r l^r \\ &\geq l - \leq \frac{1-\delta}{3-\delta} l^2.\end{aligned}$$

□

Theorem 3.2. *If* $\varphi \in \mathcal{UCVT}(\delta, \mathcal{I}_\alpha^\sigma)$ *and* $\mid z \mid= l < 1$, *then*

$$l - \frac{1-\delta}{(\alpha+1)(3-\delta)} l^2 \quad \leq \quad \mid \mathcal{I}_\alpha^\sigma \varphi(z) \mid \quad \leq \quad l + \frac{1-\delta}{(\alpha+1)(3-\delta)} l^2.$$

$$\mathcal{I}_\alpha^\sigma \varphi(z) = z - \frac{1-\delta}{(\alpha+1)(3-\delta)} z^2. \tag{18}$$

Remark 3.3. *For* $\alpha = 1$, *Theorem 3.1 and Theorem 3.2 were discussed in [9].*

By taking $\sigma = 1$ in Theorem 3.1 and Theorem 3.2, we can deduce the following:

Corollary 3.4. *If* $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha)$ *and* $\mid z \mid= l < 1$, *then*

$$l - \frac{1-\delta}{3-\delta} l^2 \quad \leq \quad \mid \mathcal{I}_\alpha(\varphi(z)) \mid \quad \leq \quad l + \frac{1-\delta}{3-\delta} l^2.$$

$$\mathcal{I}_\alpha \varphi(z) = z - \frac{1-\delta}{3-\delta} z^2. \tag{19}$$

Corollary 3.5. *If* $\varphi \in \mathcal{UCVT}(\delta, \mathcal{I}_\alpha)$ *and* $| z |= l < 1$, *then*

$$l - \frac{1-\delta}{(\alpha+1)(3-\delta)} l^2 \quad \leq \quad | \mathcal{I}_\alpha \varphi(z) | \quad \leq \quad l + \frac{1-\delta}{(\alpha+1)(3-\delta)} l^2.$$

$$\mathcal{I}_\alpha \varphi(z) = z - \frac{1-\delta}{(\alpha+1)(3-\delta)} z^2. \tag{20}$$

Theorem 3.6. *If* $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha^\sigma)$, *then*

$$1 - \frac{2\,(1-\delta)}{3-\delta} l \quad \leq \quad | (\mathcal{I}_\alpha^\sigma \varphi(z))' | \quad \leq \quad 1 + \frac{2\,(1-\delta)}{3-\delta} l. \tag{21}$$

Proof.

$$\mathcal{I}_\alpha^\sigma \varphi(z) = z - \sum_{r=2}^{\infty} \left(\frac{\alpha+1}{\alpha+r} \right)^\sigma a_r z^r$$

$$| (\mathcal{I}_\alpha^\sigma \varphi(z))' | \quad \leq \quad 1 + \sum_{r=2}^{\infty} r a_r \,| z |^{r-1} \quad \leq \quad 1 + l \sum_{r=2}^{\infty} r a_r \tag{22}$$

and

$$| (\mathcal{I}_\alpha^\sigma \varphi(z))' | \quad \geq \quad 1 - \sum_{r=2}^{\infty} r a_r \,| z |^{r-1} \quad \leq \quad 1 + l \sum_{r=2}^{\infty} r a_r. \tag{23}$$

The proof of the Theorem 3.6 would now follows from (22) and (23) by means of a rather simple consequence of (21). □

Theorem 3.7. *If* $\varphi \in \mathcal{UCVT}(\delta, \mathcal{I}_\alpha^\sigma)$ *and* $| z |= l < 1$, *then*

$$1 - \frac{2\,(1-\delta)}{(\alpha+1)(3-\delta)} l \quad \leq \quad | (\mathcal{I}_\alpha^\sigma \varphi(z))' | \quad \leq \quad 1 + \frac{2\,(1-\delta)}{(\alpha+1)(3-\delta)} l. \tag{24}$$

Proof. Proof of the Theorem 3.7 is same as the proof of Theorem 3.6 and it will be omitted. □

By taking $\alpha = 1$ in Theorem 3.6 and Theorem 3.7, we can deduce the following:

Corollary 3.8. *If* $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}^\sigma)$, *then*

$$1 - \frac{2\,(1-\delta)}{3-\delta} l \quad \leq \quad | (\mathcal{I}^\sigma \varphi(z))' | \quad \leq \quad 1 + \frac{2\,(1-\delta)}{3-\delta} l.$$

$$\mathcal{I}^\sigma \varphi(z) = z - \frac{1-\delta}{3-\delta} z^2.$$

Corollary 3.9. *If* $\varphi \in \mathcal{UCVT}(\delta, \mathcal{I}^\sigma)$ *and* $|\, z \,| = l < 1$, *then*

$$1 - \frac{(1-\delta)}{(3-\delta)} l \quad \leq \quad |\, (\mathcal{I}^\sigma \varphi(z))' \,| \quad \leq \quad 1 + \frac{(1-\delta)}{(3-\delta)} l.$$

$$\mathcal{I}^\sigma \varphi(z) = z - \frac{1-\delta}{2(3-\delta)} z^2.$$

By taking $\sigma = 1$ in Theorem 3.6 and Theorem 3.7, we can deduce the following:

Corollary 3.10. *If* $\varphi \in \mathcal{S}_p\mathcal{T}(\delta, \mathcal{I}_\alpha)$, *then*

$$1 - \frac{2\,(1-\delta)}{3-\delta} l \quad \leq \quad |\, (\mathcal{I}_\alpha \varphi(z))' \,| \quad \leq \quad 1 + \frac{2\,(1-\delta)}{3-\delta} l.$$

$$\mathcal{I}_\alpha \varphi(z) = z - \frac{1-\delta}{3-\delta} z^2.$$

Corollary 3.11. *If* $\varphi \in \mathcal{UCVT}(\delta, \mathcal{I}_\alpha)$ *and* $|\, z \,| = l < 1$, *then*

$$1 - \frac{2\,(1-\delta)}{(\alpha+1)(3-\delta)} l \quad \leq \quad |\, (\mathcal{I}_\alpha \varphi(z))' \,| \quad \leq \quad 1 + \frac{2\,(1-\delta)}{(\alpha+1)(3-\delta)} l.$$

$$\mathcal{I}_\alpha \varphi(z) = z - \frac{1-\delta}{(\alpha+1)(3-\delta)} z^2.$$

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